Inflationary Dynamics with Two Scalar Fields and Generalized Potentials

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The evolution of two slow–rolling scalar fields with potentials of the form $V = V_0\phi^{-\alpha}\exp(-\beta\phi^m)$ is studied. Considering different values of the parameters α, β and m, we derive several inflationary solutions in which both fields are dynamically important during inflation. We also discuss the evolution of perturbations in both scalar fields and the spacetime metric, concentrating on the production of entropy perturbations between both fields.

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I. INTRODUCTION

Since their advent just over twenty years ago, cosmological inflationary models have been developed to try to deal with several problems arising in the standard cosmology. The most important consequence of an inflationary stage in the very early universe is the development of perturbations in space–time and matter, which eventually evolve into the structures we observe in the universe today. The simplest model with an inflationary stage is where inflation is driven by a single scalar field, called the inflaton, with some potential $V(\phi)$. If the potential is flat enough, theory predicts that adiabatic perturbations are generated which obey the gaussian statistics and have an almost scale-free spectrum (for recent reviews see e.g. [1] and [2]).

However, the most important drawback for inflationary cosmology is that there is no unique candiate for the inflaton field and its potential. Furthermore, according to our theories of particle physics, there are a large number of scalar fields which, potentially, could be important in the early universe. For example, hybrid inflation is a model, in which two scalar fields are important: whereas one field drives an inflationary epoch, the dynamics of the second field will end this period of inflation [3]. If inflation itself is driven by two or more scalar fields, the perturbations are no longer purely adiabatic or gaussian [4]-[8]. Moreover, the transition to the normal radiation–dominated epoch depends on the decay of the scalar fields into radiation and matter, which could influence the evolution of perturbations as well. Cosmological observations by the Microwave Anisotropy Probe (MAP), the Planck Surveyor, 2dF and SLOAN Digital Sky Survey will put strong constraints on the nature of the primordial perturbations and therefore put constraints on more complicated models of inflation and the process of reheating.

In this paper, we investigate the evolution of two scalar fields, each with a potential of the form $V(\phi) = V_0\phi^{-\alpha}\exp[-\beta\phi^m]$, and examine the effect of the parameters α, β, m on the evolution of the universe. This potential includes pure exponentials ($\alpha = 0$), pure power law ($\beta = 0$) and combinations of the two. For a single scalar field, this potential was investigated in detail in [9] and [10]. Exponential potentials appear in Kaluza–Klein theories as well as in supergravity and superstring models (for a review, see e.g. [11]). Models based on dynamical supersymmetry breaking involving fermion condensates motivate inverse powerlaw potentials [12]; supergravity corrections to these models in turn predict potentials of the form above [13]. The potentials just described were used both in inflationary cosmology as well as in models for dark energy, driving the observed acceleration of the universe at the present epoch (for a review, see [14]). Indeed, the potential above has interesting properties, allowing for scaling behaviour and other attractor–like solutions (see e.g. [15] and [16]). However, our goal in this paper is to obtain an understanding of the dynamics of two scalar fields with this potential which drive an inflationary epoch in the *early* universe. This, in turn, allows us to study further the consequences for the perturbations, in the two scalar fields and the space–time metric, *during* the inflationary epoch. This can be seen as a first step to understanding the initial perturbations in the radiation dominated era, which eventually evolve into the structures we observe today. We would like to emphasize, however, that in order to calculate the perturbations in the radiation dominated epoch, an understanding of the decay of both scalar fields is needed. There are some possiblities we would like to mention:

- Both scalar fields decay, one into radiation and baryonic matter, the other field only into dark matter. The second field could, in principle, also decay only partly into dark matter, whereas the "rest" provides an explanation for dark energy.
- Only one scalar field decays completely into matter and radiation, the other decays partly into some form of matter; the remains of the field play the role of dark energy.

- One of the fields plays the role of the curvaton field [17], i.e. it decays well after the inflationary epoch and is responsible for the curvature perturbation.

There are more possibilities, of course, but it is clear that the subsequent evolution of perturbations in the metric and matter fields strongly depends on which of these possibilities is realized. The situation is now far more complicated when compared to inflationary models based on one scalar field alone.

The paper is organized as follows: in Section 2 we present the equations of motion; section 3 discusses the case of two independent potentials for both fields, i.e. the total potential is given by $V = V_0\phi^{-\alpha}\exp[-\beta\phi^m] + V_1\chi^{-\alpha}\exp[-\beta\chi^m]$; in Section 4 we investigate a mixed potential of the form $V = V_0\chi^{-\alpha}\exp[-\beta\phi^m]$. In sections 3 and 4 we discuss all of the possible cases in detail. These two sections are necessarily technical. However, we have included a summary of our results at the end of each section. Readers who are not interested in the technical aspects may wish to refer to these summaries. The perturbations are discussed in Section 5; we present our conclusions in Section 6.

II. THE FIELD DYNAMICS

Our starting point shall be that of Einstein gravity with two scalar fields and a general potential $V(\phi,\chi)$. The action for this[1] is described by

$$\mathcal{S} = \int \mathrm{d^4x}\sqrt{-\mathrm{g}}\left[\frac{1}{2}\mathrm{R} - \frac{1}{2}(\partial_\mathrm{a}\phi\partial^\mathrm{a}\phi + \partial_\mathrm{a}\chi\partial^\mathrm{a}\chi) - \mathrm{V}(\phi,\chi)\right]. \tag{1}$$

This action leads to the usual Friedmann equation and two Klein–Gordon equations for both scalar fields:

$$\begin{aligned} H^2 &= \frac{1}{3}\left[\frac{1}{2}\left(\dot\phi^2 + \dot\chi^2\right) + V(\phi,\chi)\right], \\ \ddot\phi &= -3H\dot\phi - V_\phi, \\ \ddot\chi &= -3H\dot\chi - V_\chi, \end{aligned} \tag{2}$$

where we use the notation $V_\phi = \partial V/\partial\phi$ with its obvious extensions.

We shall find it convenient to make the slow–roll approximation which is appropriate when the slow–roll parameters

$$\begin{aligned} \epsilon_\phi = \frac{1}{2}\left(\frac{V_\phi}{V}\right)^2, \quad \epsilon_\chi = \frac{1}{2}\left(\frac{V_\chi}{V}\right)^2, \\ \eta_{\phi\phi} = \frac{V_{\phi\phi}}{V}, \quad \eta_{\phi\chi} = \frac{V_{\phi\chi}}{V}, \quad \eta_{\chi\chi} = \frac{V_{\chi\chi}}{V}. \end{aligned} \tag{3}$$

satisfy $|\epsilon_i, \eta_{ij}| \ll 1, i, j = \phi, \chi$. These five conditions are sufficient for us to generate an inflationary epoch where $\ddot a(t) > 0$. With this approximation the equations of motions (2) reduce to

$$H^2 = \frac{1}{3}V(\phi,\chi), \tag{4}$$

$$3H\dot\phi = -V_\phi, \tag{5}$$

$$3H\dot\chi = -V_\chi. \tag{6}$$

For the most part, we shall see that slow–roll is an attractor in our model and that we shall be justified in making this approximation.

III. UNCOUPLED POTENTIALS

In this section we begin our investigation by considering two uncoupled potentials of the form

$$V(\phi,\chi) = V_{0_\phi}\phi^{-\alpha_1}\exp\left[-\beta_1\phi^{m_1}\right] + V_{0_\chi}\chi^{-\alpha_2}\exp\left[-\beta_2\chi^{m_2}\right]. \tag{7}$$

[1] In Planck units.

In this case the slow–roll parameters read

$$\epsilon_\phi = \frac{1}{2}\left(\frac{\alpha_1}{\phi} + \beta_1 m_1 \phi^{m_1-1}\right)^2,$$

$$\eta_{\phi\phi} = \left(\frac{\alpha_1}{\phi} + \beta_1 m_1 \phi^{m_1-1}\right)^2 + \frac{\alpha_1}{\phi^2} + \beta_1^2 m_1 (m_1-1)\phi^{m_1-2}, \tag{8}$$

and similarly for χ, with $\eta_{\phi\chi} = 0$. It is clear that, with a judicious choice of initial conditions, these can always be made small. It is also clear that $m_i = 1$ will be a transitional value as for $m_i > 1$ the slow–roll parameters will grow. We now begin to examine the evolution of the fields in this regime. We shall assume that $\beta_1 > 0$ but make no other assumptions about the other parameters in the potential terms.

Large Field Values

A. $m_1, m_2 > 0$

With $m_1, m_2 > 0$, the exponential part of the potential, for large enough ϕ, χ at least, will always dominate over the power law. Dividing equations (5-6) we generate the equation

$$\frac{\dot\phi}{V_\phi} = \frac{\dot\chi}{V_\chi} \tag{9}$$

We are unable to integrate this directly but are able to do so to leading order in the fields by means of integration by parts. This then generates for large ϕ, χ

$$\beta_1^2 m_1^2 \phi^{2m_1-2} V_{0_\phi} \phi^{-\alpha_1} \exp\left[-\beta_1 \phi^{m_1}\right] = \beta_2^2 m_2^2 \chi^{2m_2-2} V_{0_\chi} \chi^{-\alpha_2} \exp\left[-\beta_2 \chi^{m_2}\right]. \tag{10}$$

One then takes natural logarithms to find a simple expression between the fields. Since $m_i > 0$ the terms proportional to the powers of the fields will eventually dominate. This gives us

$$\beta_1 \phi^{m_1} = \beta_2 \chi^{m_2}, \tag{11}$$

to leading order. Then using (4) and (11) we are able to show

$$H(\phi) = \left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \phi^{-\frac{\alpha_1}{2}} \exp\left[\frac{-\beta_1\phi^{m_1}}{2}\right] \left(1 + \frac{m_1^2}{m_2^2}\left(\frac{\beta_1}{\beta_2}\right)^{\frac{2}{m_2}} \phi^{\frac{2m_1}{m_2}-2}\right)^{\frac{1}{2}}. \tag{12}$$

Substituting back into (5) we must solve the differential equation for ϕ

$$\dot\phi = \left(\frac{V_{0\phi}}{3}\right)^{\frac{1}{2}} \frac{\frac{\alpha_1}{\phi} + \beta_1 m_1 \phi^{m_1-1}}{\left(1 + \frac{m_1^2}{m_2^2}\left(\frac{\beta_1}{\beta_2}\right)^{\frac{2}{m_2}} \phi^{\frac{2m_1}{m_2}-2}\right)^{\frac{1}{2}}} \phi^{-\frac{\alpha_1}{2}} \exp\left[\frac{-\beta_1\phi^{m_1}}{2}\right], \tag{13}$$

$$= \left(\frac{V_{0\phi}}{3}\right)^{\frac{1}{2}} \frac{\beta_1 m_1 \phi^{-\frac{\alpha_1}{2}+m_1-1}}{\left(1 + \frac{m_1^2}{m_2^2}\left(\frac{\beta_1}{\beta_2}\right)^{\frac{2}{m_2}} \phi^{\frac{2m_1}{m_2}-2}\right)^{\frac{1}{2}}} \exp\left[\frac{-\beta_1\phi^{m_1}}{2}\right], \tag{14}$$

to leading order in ϕ. For large ϕ, we are able to integrate this expression under two different regimes, depending on whether $m_1 < m_2$ or $m_1 = m_2$. This corresponds to one field giving the dominant contribution to the Hubble parameter and both giving equal contributions. This is clear from the denomenator in (14). We shall take both of these cases in turn.

1. $m_1 < m_2$

In this case it is easy to see that the ϕ field is the dominant contributor. Solving for ϕ we integrate (14) by parts to show

$$t(\phi) = \left(\frac{3}{V_{0_\phi}}\right)^{\frac{1}{2}} \frac{2}{\beta_1^2 m_1^2} \phi^{\frac{\alpha_1}{2} - 2m_1 + 2} \exp\left[\frac{\beta_1 \phi_1^m}{2}\right]. \tag{15}$$

Immediately we are able to see that large ϕ implies large t and so inflation will occur at late times with this set up. We are unable to invert this directly. However by taking ln, we see that the exponential quickly dominates, and so to leading order

$$\phi(t) = \left[\frac{2}{\beta_1} \ln\left[\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \frac{\beta_1^2 m_1^2}{2} t\right]\right]^{\frac{1}{m_1}}. \tag{16}$$

Writing (15) in the form

$$\frac{\beta_1 \phi^{m_1}}{2} = \ln\left[\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \frac{\beta_1^2 m_1^2}{2} \phi^{-\frac{\alpha_1}{2} + 2m_1 - 2} t\right], \tag{17}$$

we are able to substitute back into (16) to obtain a solution for $\phi(t)$ to next order in t. A little thought reveals why we want and need to do this. When we come to evaluate the Hubble parameter as a function of time we need to take the exponent of ϕ. Every time we do this we must take the next to leading order expression. This is given by

$$\phi(t) = \left[\frac{2}{\beta_1} \ln\left[\frac{\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \frac{\beta_1^2 m_1^2}{2} t}{\left[\frac{2}{\beta_1} \ln\left[\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \frac{\beta_1^2 m_1^2}{2} t\right]\right]^{\frac{4+\alpha_1}{2m_1} - 2}}\right]\right]^{\frac{1}{m_1}}. \tag{18}$$

Then we plug this into (12) to solve for $H(t)$ and then integrate to get the scalefactor which gives us,

$$H(t) = \frac{1}{\beta_1 m_1^2} \left(\frac{2}{\beta_1}\right)^{\frac{2-m_1}{m_1}} \frac{1}{t} \left[\ln t\right]^{\frac{2-m_1}{m_1}}, \tag{19}$$

$$a(t) \propto \exp\left[\frac{1}{\beta_1 m_1 (2 - m_1)} \left[\frac{2}{\beta_1} \ln t\right]^{\frac{2-m_1}{m_1}}\right]. \tag{20}$$

In the evolution of the fields we may drop most of the coefficients as they simply result in constant terms in the solution. This gives the final solution

$$\phi(t) = \left(\frac{2}{\beta_1} \ln t\right)^{\frac{1}{m_1}}, \tag{21}$$

$$\chi(t) = \left(\frac{2}{\beta_2} \ln t\right)^{\frac{1}{m_2}}. \tag{22}$$

Now because $m_i < 1$, we see that the scale factor grows faster than power law. This regime is effectively equivalent to a single scalar field driving the dynamics of the universe as the second scalar χ sits in the background. Unsurprisingly, the solutions are in direct agreement with section III.A of [10].

2. $m_1 > m_2$

This is equivalent to the previous section by symmetry.

3. $m_1 = m_2 = m$

We now consider the fields to give equal contributions to the evolution of our universe. The second field is now significant so we expect to see something new. This case includes that of two straight exponential potentials which has already been studied in [18]. We able to integrate (14) to leading order to show that

$$t(\phi) = \left(\frac{3}{V_{0_\phi}}\right)^{\frac{1}{2}} \frac{2\left(1+\left(\frac{\beta_1}{\beta_2}\right)^{\frac{2}{m}}\right)^{\frac{1}{2}}}{\beta_1^2 m^2} \phi^{\frac{\alpha_1}{2}-2m+2} \exp\left[\frac{\beta_1 \phi^m}{2}\right]. \tag{23}$$

Once more the evolution of $t(\phi)$ demonstrates that inflation occurs at late times for large ϕ, χ. We then proceed in the same way and the solutions we get are as follows:

$$\phi(t) = \left(\frac{2}{\beta_1}\ln t\right)^{\frac{1}{m}}, \tag{24}$$

$$\chi(t) = \left(\frac{2}{\beta_2}\ln t\right)^{\frac{1}{m}}, \tag{25}$$

$$H(t) = \left(1+\left(\frac{\beta_1}{\beta_2}\right)^{\frac{2}{m}}\right) \frac{1}{\beta_1 m^2}\left(\frac{2}{\beta_1}\right)^{\frac{2-m}{m}} \frac{1}{t}\left[\ln t\right]^{\frac{2-2m}{m}}, \tag{26}$$

$$a(t) \sim \exp\left[\left(1+\left(\frac{\beta_1}{\beta_2}\right)^{\frac{2}{m}}\right) \frac{1}{\beta_1 m(2-m)}\left[\frac{2}{\beta_1}\ln t\right]^{\frac{2-m}{m}}\right]. \tag{27}$$

It should be noted that in the two exponential potential case this reduces to power law inflation

$$a(t) \sim t^{\frac{2(\beta_1^2+\beta_2^2)}{\beta_1^2\beta_2^2}}, \tag{28}$$

which is agreement with the dynamical system analysis of [18]. The "new" feature is that we generate more inflation than we would get with one scalar because of the extra factor, $\exp\left[1+\left(\frac{\beta_1}{\beta_2}\right)^{\frac{2}{m}}\right]$, in the solution for the scale factor. This concludes the analysis for the cases $m_1, m_2 > 0$.

B. $m_1 \leq 0, m_2 > 0$

It is possible when $m_i < 0$ and is also an even number, to generate a minimum of the potential at the origin. The nature of the slow-roll parameters dictates that this will certainly give a finite amount of inflation and that we may not get slow-roll here. We shall assume, initially at least, that the field begins in the part dominated by the power law part of the potential.

1. $\alpha_1 \neq 0, -4$

Following the same procedure as the previous section we are able to deduce that

$$\dot{\phi} = \left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \alpha_1 \phi^{-\frac{\alpha_1+2}{2}} \exp\left[-\frac{\beta_1 \phi^{m_1}}{2}\right]. \tag{29}$$

Taking the ϕ exponential as approximately constant, we are able to solve this to give,

$$t(\phi) = \left(\frac{3}{V_{0_\phi}}\right)^{\frac{1}{2}} \frac{2}{\alpha_1(\alpha_1+4)} \phi^{\frac{\alpha_1+4}{2}} \exp\left[\frac{\beta_1 \phi^{m_1}}{2}\right]. \tag{30}$$

Solving for the remaining fields and parameters we deduce

$$\phi(t) = \left(\left(\frac{V_{0_\phi}}{3} \right)^{\frac{1}{2}} \frac{\alpha_1}{2} (\alpha_1 + 4) t \right)^{\frac{2}{\alpha_1+4}}, \tag{31}$$

$$\chi(t) = \left[\frac{2}{\beta_2} \frac{\alpha_1 + 2}{\alpha_1 + 4} \ln t \right]^{\frac{1}{m_2}}, \tag{32}$$

$$H(t) = \left(\frac{V_{0_\phi}}{3} \right)^{\frac{1}{2}} \left(\left(\frac{V_{0_\phi}}{3} \right)^{\frac{1}{2}} \frac{\alpha_1}{2} (\alpha_1 + 4) t \right)^{-\frac{\alpha_1}{\alpha_1+4}} \times \exp \left[-\frac{\beta_1}{2} \left(\left(\frac{V_{0_\phi}}{3} \right)^{\frac{1}{2}} \frac{\alpha_1}{2} (\alpha_1 + 4) t \right)^{\frac{2m_1}{\alpha_1+4}} \right], \tag{33}$$

$$a(t) \propto \exp \left[\frac{4 + \alpha_1}{4} \left(\frac{V_{0_\phi}}{3} \right)^{\frac{1}{2}} \left(\left(\frac{V_{0_\phi}}{3} \right)^{\frac{1}{2}} \frac{\alpha_1}{2} (\alpha_1 + 4) \right)^{-\frac{\alpha_1}{\alpha_1+4}} t^{\frac{4}{\alpha_1+4}} f_1(t) \right]. \tag{34}$$

In this instance

$$f_1(t) = \exp \left[-\frac{\beta_1}{2} \left(\left(\frac{V_{0_\phi}}{3} \right)^{\frac{1}{2}} \frac{\alpha_1}{2} (\alpha_1 + 4) t \right)^{\frac{2m_1}{\alpha_1+4}} \right], \tag{35}$$

and rapidly tends to unity as time increases. With this set of parameters, studying (30) one realises that it is possible for inflation to occur in a different manner. It is helpful to write the scalefactor in the form

$$a(t) \propto \exp \left[c t^k f_1(t) \right] \tag{36}$$

$\alpha_1 > 0$: In this case we have $\alpha_1(\alpha_1 + 4) > 0$. A moments thought reveals that $t(\phi) \to \infty$ as ϕ increases. Therefore we must generate inflation at late times. This also gives us $c > 0, 0 < k < 1$ and so this gives us a relatively slow expansion rate.
$-4 < \alpha_1 < 0$: In this case $\alpha_1(\alpha_1 + 4) < 0$ and we get $c > 0$, $1 < k < \infty$. As $\phi \to \infty$, $t(\phi) \to -\infty$ and so inflation now occurs at early times. Potentially this can also give us a lot of inflation as the scale factor grows faster than e^t.
$\alpha_1 < 0$: Once more we get $\alpha_1(\alpha_1 + 4) > 0$, however we now have a negative power of ϕ in (30) and so $t(\phi) \to 0$ as ϕ increases. This generates inflation at early times once more. In addition we have the constraints $c, k > 0$.

Once more the setup is equivalent to that of a single scalar as the second scalar makes minimal contribution to the universal dynamics. We generate identical results to section III.B of [10].

Let us now return to the case where χ dominates. This gives a Hubble parameter of

$$H(\phi) = \left(\frac{V_{0_\phi} \alpha_1 (\alpha_1 + 2)}{3 \beta_2^2 m_2^2} \right)^{\frac{1}{2}} \left(\frac{\alpha_1 + 2}{\beta_2} \right)^{\frac{1-m_2}{m_2}} \phi^{-\frac{(\alpha_1+2)}{2}} \left[\ln \phi \right]^{\frac{1}{m_2} - 1} \exp \left[-\frac{\beta_1 \phi^{m_1}}{2} \right], \tag{37}$$

$$\dot{\phi} = \left(\frac{V_{0_\phi} \alpha_1 \beta_2^2 m_2^2}{3 (\alpha_1 + 2)} \right)^{\frac{1}{2}} \left(\frac{\alpha_1 + 2}{\beta_2} \right)^{\frac{1-m_2}{m_2}} \phi^{-\frac{\alpha_1}{2}} \left[\ln \phi \right]^{1 - \frac{1}{m_2}} \exp \left[-\frac{\beta_1 \phi^{m_1}}{2} \right]. \tag{38}$$

Integrating up, again regarding the exponential as almost constant we get

$$t(\phi) = \left(\frac{3 (\alpha_1 + 2)}{V_{0_\phi} \alpha_1 \beta_2^2} \right)^{\frac{1}{2}} \left(\frac{\alpha_1 + 2}{\beta_2} \right)^{\frac{m_2-1}{m_2}} \phi^{\frac{\alpha_1-2}{2}} \left[\ln \phi \right]^{\frac{1}{m_2}} \exp \left[\frac{\beta_1 \phi^{m_1}}{2} \right]. \tag{39}$$

We are not able to invert this equation in order to obtain a meaningful description of $\phi(t)$.

2. $\alpha_1 = -4$

The evolution equation for ϕ (29) still holds – provided, of course, that we are still in a ϕ dominated regime.

$$\dot{\phi} = -4\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\phi\exp\left[-\frac{\beta_1\phi^{m_1}}{2}\right]. \tag{40}$$

Then integrating we retrieve

$$t(\phi) = -\frac{1}{4}\left(\frac{3}{V_{0_\phi}}\right)^{\frac{1}{2}}(\ln\phi)\exp\left[\frac{\beta_1\phi^{m_1}}{2}\right]. \tag{41}$$

Notice that we generate a ln rather than a power of ϕ in the solution. This propagates through the calculation and will generate qualitively different behaviour in the Hubble parameter and scalefactor. However, we still see that $t(\phi) \to -\infty$ for increasing ϕ and so it is an early effect.

$$\phi(t) = \exp\left[-4\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}t\right], \tag{42}$$

$$\chi(t) = \left[\frac{8}{\beta_2}\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}t\right]^{\frac{1}{m_2}}, \tag{43}$$

$$H(t) = \left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\exp\left[-8\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}t - \frac{\beta_1}{2}\exp\left[-4m_1\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}t\right]\right], \tag{44}$$

$$a(t) \propto \exp\left[-\frac{1}{8}\exp\left[-8\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}t\right]f_3(t)\right], \tag{45}$$

where

$$f_3(t) = \exp\left[-\frac{\beta_1}{2}\exp\left[-4m_1\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}t\right]\right]. \tag{46}$$

It is clear that $f_3(t) \to 1$ rapidly as t increases. Looking at equation (41), it is clear that t decreases for increasing ϕ so the inflation we generate is an early time phenomenon.

3. $\alpha_1 = 0$

In this special case the ϕ field will quickly dominate once more, as the potential for χ decays quickly. Thus we get behaviour very similar to that for one field, with χ just in the background as in [10]. Solving in the usual manner gives us

$$\phi(t) = \left[\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\beta_1 m_1(2-m_1)\right]^{\frac{1}{2-m_1}} t^{\frac{1}{2-m_1}}, \tag{47}$$

$$\chi(t) = \left[\frac{1}{\beta_2}\ln t\right]^{\frac{1}{m_2}}, \tag{48}$$

$$H(t) = \left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\exp\left[-\frac{\beta_1}{2}\left(\beta_1 m_1(2-m_1)\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\right)^{\frac{m_1}{2-m_1}} t^{\frac{m_1}{2-m_1}}\right], \tag{49}$$

$$a(t) \propto \exp\left[\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} t f_2(t)\right], \tag{50}$$

where

$$f_2(t) = \exp\left[-\frac{\beta_1}{2}\left(\beta_1 m_1(2-m_1)\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\right)^{\frac{m_1}{2-m_1}} t^{\frac{m_1}{2-m_1}}\right]. \tag{51}$$

Inflation here occurs at early times.

C. $m_1, m_2 < 0$

Proceeding in the usual manner we are able to show that

$$H(\phi) = \left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \phi^{-\frac{\alpha_1}{2}} \exp\left[-\frac{\beta_1\phi^{m_1}}{2}\right] \left(1 + \frac{\alpha_1(\alpha_1+2)}{\alpha_2(\alpha_2+2)}\frac{\chi^2}{\phi^2}\right)^{\frac{1}{2}}, \tag{52}$$

$$= \left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \phi^{-\frac{\alpha_1}{2}} \exp\left[-\frac{\beta_1\phi^{m_1}}{2}\right] \times \left(1 + \frac{\alpha_1(\alpha_1+2)}{\alpha_2(\alpha_2+2)}\left(\frac{V_{0_\chi}\alpha_2(\alpha_2+2)}{V_{0_\phi}\alpha_1(\alpha_1+2)}\right)^{\frac{2}{\alpha_2+2}} \phi^{\frac{2(\alpha_1-\alpha_2)}{\alpha_2+2}}\right)^{\frac{1}{2}}. \tag{53}$$

It is then clear, for example, that $-2 < \alpha_1 < \alpha_2$ corresponds to ϕ domination. We shall consider this case and that of equal contribution since χ domination is identical in behaviour to the former.

1. $\alpha_2 > \alpha_1 > -2; \alpha_2 < -2, \alpha_1 > \alpha_2$

If is clear that the ϕ field is dominant here and so the results from section (III B 1) hold for $\phi(t), H(t)$ and $a(t)$. In this case we do get different evolution for $\chi(t)$ though, given by

$$\chi(t) = \left(\frac{V_{0_\chi}\alpha_2(\alpha_2+2)}{V_{0_\phi}\alpha_1(\alpha_1+2)}\right)^{\frac{1}{\alpha_2+2}} \left[\left(\frac{V_{0_\phi}}{3}\right)\frac{\alpha_1}{2}(\alpha_1+4)t\right]^{\frac{2(\alpha_1+2)}{(\alpha_1+4)(\alpha_2+2)}}. \tag{54}$$

2. $\alpha_1 = \alpha_2 = \alpha$

This is the only case where we do not get domination by one field over the other. Again the results will be very similar to that of section (III B 1) except that we will generate more inflation because the Hubble parameter will be larger than before. Then we find that:

$$t(\phi) = \left(\frac{3}{V_{0_\phi}}\right)^{\frac{1}{2}} \frac{2\left(1+\left(\frac{V_{0_\chi}}{V_{0_\phi}}\right)^{\frac{2}{\alpha+2}}\right)^{\frac{1}{2}}}{\alpha(\alpha+4)} \phi^{\frac{\alpha+4}{2}} \exp\left[\frac{\beta_1\phi^{m_1}}{2}\right], \tag{55}$$

$$\phi(t) = \left(\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\frac{\alpha}{2}(\alpha+4)\left(1+\left(\frac{V_{0_\chi}}{V_{0_\phi}}\right)^{\frac{2}{\alpha+2}}\right)^{-\frac{1}{2}} t\right)^{\frac{2}{\alpha+4}}, \tag{56}$$

$$\chi(t) = \left(\frac{V_{0_\chi}}{V_{0_\phi}}\right)^{\frac{1}{\alpha+2}} \left(\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\frac{\alpha}{2}(\alpha+4)\left(1+\left(\frac{V_{0_\chi}}{V_{0_\phi}}\right)^{\frac{2}{\alpha+2}}\right)^{-\frac{1}{2}} t\right)^{\frac{2}{\alpha+4}}, \tag{57}$$

$$H(t) = \left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \left(1+\left(\frac{V_{0_\chi}}{V_{0_\phi}}\right)^{\frac{2}{\alpha+2}}\right)^{\frac{\alpha+2}{\alpha+4}} \left(\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\frac{\alpha}{2}(\alpha+4)t\right)^{-\frac{\alpha}{\alpha+4}} \times \exp\left[-\frac{\beta_1}{2}\left(\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\frac{\alpha}{2}(\alpha+4)\left(1+\left(\frac{V_{0_\chi}}{V_{0_\phi}}\right)^{\frac{2}{\alpha+2}}\right)^{-\frac{1}{2}} t\right)^{\frac{2m_1}{\alpha+4}}\right], \tag{58}$$

$$a(t) \propto \exp\left[\frac{4+\alpha}{4}\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \left(1+\left(\frac{V_{0_\chi}}{V_{0_\phi}}\right)^{\frac{2}{\alpha+2}}\right)^{\frac{\alpha+2}{\alpha+4}} \left(\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\frac{\alpha}{2}(\alpha+4)\right)^{-\frac{\alpha}{\alpha+4}} t^{\frac{4}{\alpha+4}} f_4(t)\right]. \tag{59}$$

where

$$f_4(t) = \exp\left[-\frac{\beta_1}{2}\left(\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)\left(1+\left(\frac{V_{0_\chi}}{V_{0_\phi}}\right)^{\frac{2}{\alpha+2}}\right)^{-\frac{1}{2}} t\right)^{\frac{2m_1}{\alpha+4}} \right]. \quad (60)$$

$f_4(t) \to 1$ rapidly as t increases. This gives us inflation at late times once more.

Small Field Values

A moments thought reveals that in order to generate slow–roll at small values of either χ or ϕ we must have $\alpha_i = 0, m \geq 1$. This can be seen easily by examining the slow–roll parameters (3). Our analysis will then follow a similar pattern to before. Again, we shall see that one field will become dominant if $m_1 \neq m_2$ and we proceed as before.

D. $m_1 < m_2, m_1 \neq 2$

Once more the ϕ will provide the dominant contribution and so

$$H(\phi) = \left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \exp\left[-\frac{\beta_1 \phi^{m_1}}{2}\right]. \quad (61)$$

Then we find that

$$\dot{\phi} = \left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \beta m_1 \phi^{m_1 - 1} \exp\left[-\frac{\beta_1 \phi^{m_1}}{2}\right]. \quad (62)$$

This is identical to the case we considered previously in section (III B 3) and so all the results there still hold. The one difference is that the evolution of χ changes slightly.

$$\chi(t) = \left[\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \frac{V_{0\phi}}{V_{0\chi}} \beta_2 m_2 (2 - m_2) t\right]^{\frac{1}{m_2 - 2}}. \quad (63)$$

One should note that as $\phi \to 0$, t decreases and so we are looking at an early time effect. This is equivalent to section IV.A of [10]. The cases where either one or both of $m_i = 2$ are easily covered and does not result in any analytical difficulties. The behaviour is not markedly different from the above.

E. $2 = m_1 < m_2$

The ϕ is still dominant and so we can integrate (62) to achieve

$$t(\phi) = \left(\frac{3}{V_0}\right)^{\frac{1}{2}} \frac{1}{2\beta_1} (\ln \phi) \exp\left[-\frac{\beta_1 \phi^2}{2}\right]. \quad (64)$$

Solving the equation for small ϕ we find that

$$\phi(t) = \exp\left[2\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \beta t\right], \quad (65)$$

$$\chi(t) = \left[4\left(\frac{V_{0\phi}}{3}\right)^{\frac{1}{2}} \frac{V_{0_\phi}}{V_{0_\chi}} \frac{\beta_1^2}{\beta_2 m_2 (2 - m_2)} t\right]^{\frac{1}{m_2 - 2}}. \quad (66)$$

Note that although the behaviour $\chi(t) \sim (-\kappa_1 t)^{\kappa_2}, \kappa_1 > 0$ appears problematic, this would be resolved by including the integration constants. This would then give us

$$\chi(t) = \left[K - 4\left(\frac{V_{0\phi}}{3}\right)^{\frac{1}{2}} \frac{V_{0_\phi}}{V_{0_\chi}} \frac{\beta_1^2}{\beta_2 m_2(2-m_2)} t\right]^{\frac{1}{m_2-2}}, \quad K > 0. \tag{67}$$

Now since $\phi(t)$ in a monotonic increasing function, the slow–roll conditions will be violated within finite time and so our solution will no longer be valid. This should occur before $t = \frac{K}{4}\left(\frac{3}{V_{0\phi}}\right)^{\frac{1}{2}} \frac{V_{0_\chi}}{V_{0_\phi}} \frac{\beta_2 m_2(2-m_2)}{\beta_1^2}$ since there are no physical grounds upon which to rule out this solution.

F. $m_1 = m_2 = 2$

In this instance we find that

$$H(\phi) = \frac{1}{\sqrt{3}}\left(V_{0_\phi} + V_{0_\chi}\right)^{\frac{1}{2}} \exp\left[-\frac{\beta_1 \phi^2}{2}\right]. \tag{68}$$

The it is not too difficult to show that

$$t(\phi) = \frac{\sqrt{3}}{2\beta_1 V_{0_\phi}}\left(V_{0_\phi} + V_{0_\chi}\right)^{\frac{1}{2}} (\ln\phi) \exp\left[\frac{\beta_1 \phi^2}{2}\right]. \tag{69}$$

It is then clear that as $\phi \to 0$ we have $t \to -\infty$ and so inflation will occur at early times. Then, proceeding in the familiar manner, we can show

$$\phi(t) = \exp\left[\frac{V_{0_\phi}}{\sqrt{3}} \frac{2\beta_1}{\left(V_{0_\phi} + V_{0_\chi}\right)^{\frac{1}{2}}} t\right], \tag{70}$$

$$\chi(t) = \exp\left[\frac{V_{0_\chi}}{\sqrt{3}} \frac{2\beta_2}{\left(V_{0_\phi} + V_{0_\chi}\right)^{\frac{1}{2}}} t\right], \tag{71}$$

$$H(t) = \frac{1}{\sqrt{3}}\left(V_{0_\phi} + V_{0_\chi}\right)^{\frac{1}{2}} \exp\left[-\frac{\beta_1}{2} \exp\left[\frac{V_{0_\phi}}{\sqrt{3}} \frac{2\beta_1}{\left(V_{0_\phi} + V_{0_\chi}\right)^{\frac{1}{2}}} t\right]\right], \tag{72}$$

$$a(t) \propto \exp\left[H(t)t\right]. \tag{73}$$

where we have treated all of the exponentials as begin approximately equal to unity at first approximation.

Large and small field values

If we take one of the fields large and one small we will always have the large field dominant and so this relates to the cases considered already. For example, small χ means that $m_2 \geq 1$ and so $m_1 < m_2$. Therefore the ϕ field will be dominant. This applies to all possible cases.

G. Summary for Uncoupled Potential

We have seen that for two fields with uncoupled potentials often one field will be dominant, and we return to an effective single field theory – for the background at least– as studied in [10]. There are, however, several cases where the second field is important and we have demonstrated them. We have seen that it is possible to generate inflation at both late and early times and for large and small values of both of the fields. As a result, these background solutions do not differ qualitatively from the single field set-up [10]. The results are summarized in the table below. If $m_1 < m_2$ then the ϕ field is dominant so we only outline the results for m_1. If $m_1 = m_2 = m$ the same qualitative behaviour exists except that we generate more inflation than we would normally with one field.

TABLE I. Summary of Inflationary Behaviour for uncoupled potentials

Early Times	Late Times	All Times
$m_1 \leq 0, \alpha_1 \leq 0$	$0 \leq m_1 \leq 1$	$m_1 = 1, \alpha_1 = 0$
or	or	or
$m_1 \geq 1, \alpha_1 = 0$	$m_1 \leq 0, \alpha_1 \geq 0$	$m_1 = \alpha_1 = 0$

IV. A COUPLED POTENTIAL

We now turn our attention to a potential of the form

$$V(\phi,\chi) = V_0 \phi^{-\alpha} e^{-\beta\chi^m} \tag{74}$$

and repeat the processes of the previous section. We still have the three coupled equations (4-6) to solve which are equivalent to

$$H = \left(\frac{V_0}{3}\right)^{\frac{1}{2}} \phi^{-\frac{\alpha}{2}} \exp\left[-\frac{\beta\chi^m}{2}\right], \tag{75}$$

$$3H\dot{\phi} = \alpha V_0 \phi^{-(\alpha+1)} \exp\left[-\beta\chi^m\right], \tag{76}$$

$$3H\dot{\chi} = \beta m \chi^{m-1} V_0 \phi^{-\alpha} \exp\left[-\beta\chi^m\right], \tag{77}$$

with solution

$$\phi^2 = \begin{cases} \frac{2\alpha}{\beta m(2-m)}\chi^{2-m}, & m \neq 2, \\ \frac{\alpha}{\beta}\ln\chi, & m = 2. \end{cases} \tag{78}$$

There are certain cases where it is not clear that equation (78) means anything– for example $m > 2, \alpha > 0$– since this would generate an imaginary value for at least one of the fields. At this stage one should remember that we have omitted all integration constants up this point. As there are no physical grounds on which to rule out such cases, when one puts back these constants (dependent on the initial conditions), any such problems may be overcome.

Before we begin, we shall find it instructive to examine some plots of the potential in question. These are shown in figure 1 where we have plotted $V(\phi,\chi) = \phi^{-\frac{1}{2}} \exp\left[-2\chi^{\pm 0.2}\right]$ respectively.

Large field Values

A. $1 \geq m > 0$

This guarantees us slow–roll for large values of the fields. We generate the following equation for χ

$$\dot{\chi} = \left(\frac{V_0}{3}\right)^{\frac{1}{2}} \beta m \left(\frac{\beta m(2-m)}{2\alpha}\right)^{\frac{\alpha}{4}} \chi^{\frac{\alpha(m-2)}{4}} \chi^{m-1} \exp\left[\frac{-\beta\chi^m}{2}\right]. \tag{79}$$

We are then able to solve this to produce the following expression,

$$t(\chi) = \left(\frac{3}{V_0}\right)^{\frac{1}{2}} \frac{2}{\beta^2 m^2} \left(\frac{\beta m(2-m)}{2\alpha}\right)^{-\frac{\alpha}{4}} \chi^{\frac{\alpha(2-m)}{4}} \chi^{2-2m} \exp\left[\frac{\beta\chi^m}{2}\right]. \tag{80}$$

Proceeding in a similar, way we are able to solve the system producing, to leading order in t, the results below.

$$\phi(t) = \left(\frac{2\alpha}{\beta m(2-m)}\right)^{\frac{1}{2}} \left[\frac{2}{\beta}\ln t\right]^{\frac{2-m}{2m}}, \tag{81}$$

$$\chi(t) = \left[\frac{2}{\beta}\ln t\right]^{\frac{1}{m}}, \tag{82}$$

$$H(t) = \frac{2}{\beta^2 m^2}\frac{1}{t}\left[\frac{2}{\beta}\ln t\right]^{\frac{2-2m}{m}}, \tag{83}$$

$$a(t) \propto \exp\left[\frac{1}{\beta m(2-m)}\left[\frac{2}{\beta}\ln t\right]^{\frac{2-m}{m}}\right]. \tag{84}$$

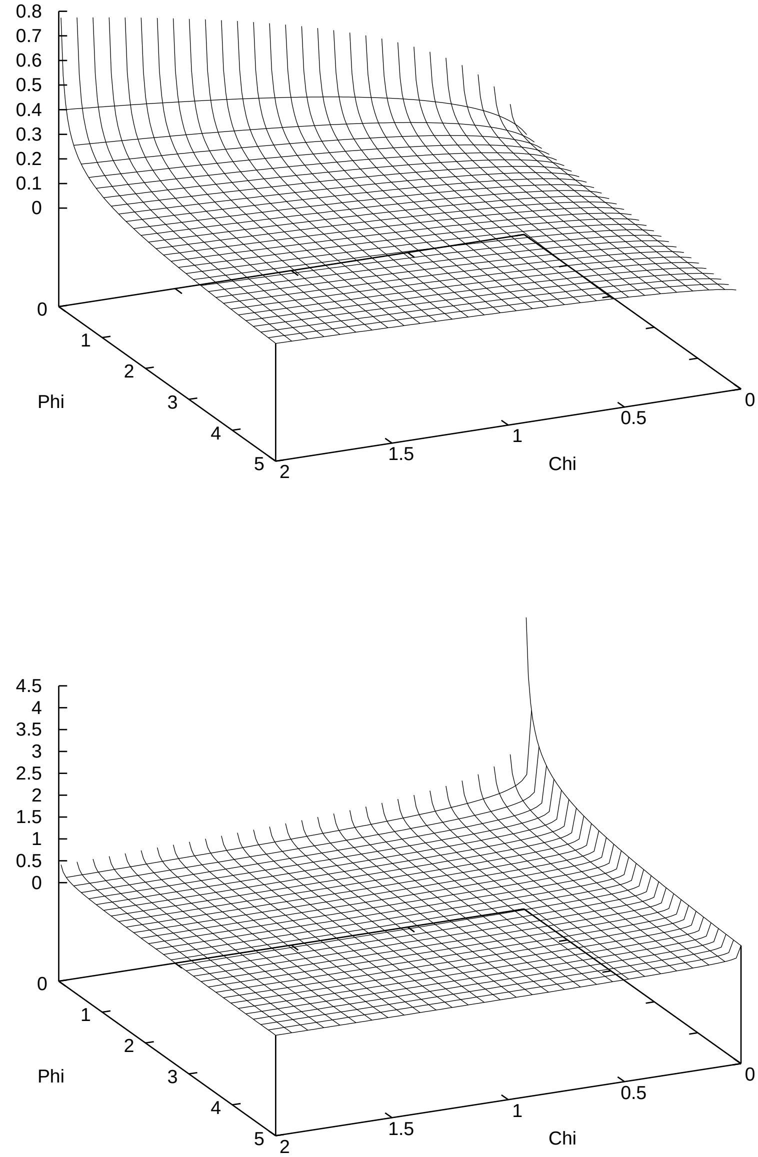

FIG. 1. Plot of the Coupled potential

One should note that this gives very similar behaviour to section (III A 1). This should not be entirely unexpected because the exponetial part of the potential will always dominate for $m > 0$.

B. $m < 0$

Due to the nature of the slow–roll parameters, we must once more consider large values for both of the fields. Now solving for ϕ, treating the exponential approximately constant, we find

$$t(\phi, \chi) = \left(\frac{3}{V_0}\right)^{\frac{1}{2}} \frac{2}{\alpha(\alpha+4)} \phi^{\frac{\alpha+4}{2}} \exp\left[\frac{\beta\chi^m}{2}\right]. \tag{85}$$

Substituting for either field using (78) we able to invert this finding

$$\chi(t) = \left(\frac{\beta m(2-m)}{2\alpha}\right)^{\frac{2}{2-m}} \left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{\frac{4}{(2-m)(\alpha+4)}}, \tag{86}$$

$$\phi(t) = \left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{\frac{2}{\alpha+4}}. \tag{87}$$

One should also note that the onset of inflation occurs in analogy with section (III B 1). It is also worth pointing out that this gives the solution one would expect when $m \to 0$. That is, we generate the same results as (III B 1) except that the χ field freezes out. This should not surprise us since the potential becomes equivalent to that of an inverse power. We then find the form for the Hubble parameter

$$H(t) = \left(\frac{V_0}{3}\right)^{\frac{1}{2}} \left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{-\frac{\alpha}{\alpha+4}}$$
$$\times \exp\left[-\frac{\beta}{2}\left(\frac{\beta m(2-m)}{2\alpha}\right)^{\frac{2m}{2-m}} \left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{\frac{4m}{(2-m)(\alpha+4)}}\right], \tag{88}$$

where, of course, the exponential part tends rapidly towards 1. Solving for the scale factor we deduce to leading order

$$a(t) \propto \exp\left[\frac{4+\alpha}{4}\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \left(\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)\right)^{-\frac{\alpha}{\alpha+4}} t^{\frac{4}{\alpha+4}} g_1(t)\right], \tag{89}$$

where

$$g_1(t) = \exp\left[-\frac{\beta}{2}\left(\frac{\beta m(2-m)}{2\alpha}\right)^{\frac{2m}{2-m}} \left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{\frac{4m}{(2-m)(\alpha+4)}}\right]. \tag{90}$$

It is obvious that the special case of $\alpha = 0$ gives us just a single scalar field that evolves. This has already been studied. We must however consider what happens when $\alpha = -4$. We generate the following equation

$$t(\phi, \chi) = -\frac{1}{4}\left(\frac{3}{V_0}\right)^{\frac{1}{2}} [\ln \phi] \exp\left[-\frac{\beta\chi^m}{2}\right]. \tag{91}$$

Then solving for the fields we find

$$\phi(t) = \exp\left[-4\left(\frac{V_0}{3}\right)^{\frac{1}{2}} t\right], \tag{92}$$

$$\chi(t) = \exp\left[-\frac{8}{2-m}\left(\frac{V_0}{3}\right)^{\frac{1}{2}} t\right], \tag{93}$$

$$H(t) = \left(\frac{V_0}{3}\right)^{\frac{1}{2}} \exp\left[-8\left(\frac{V_0}{3}\right)^{\frac{1}{2}} t - \frac{\beta}{2}\exp\left[-\frac{8m}{2-m}\left(\frac{V_0}{3}\right)^{\frac{1}{2}} t\right]\right], \tag{94}$$

$$a(t) \propto \exp\left[-\frac{1}{8}\exp\left[-8\left(\frac{V_0}{3}\right)^{\frac{1}{2}} t\right] g_2(t)\right] \tag{95}$$

where

$$g_2(t) = \exp\left[-\frac{\beta}{2}\exp\left[-\frac{8m}{2-m}\left(\frac{V_0}{3}\right)^{\frac{1}{2}} t\right]\right], \tag{96}$$

which rapidly tends towards unity.

C. $1 < m < 2$

If one considers the slow–roll parameters we may expect only to generate inflation for small values of χ. However, we must still take large ϕ for slow–roll to be a valid approximation. In this instance, we are able to approximate the exponential by unity. Provided $m \neq 2$, we generate the same set of results as for $m < 0$ with large χ, as in section IV B.

In the special case where $m = 2$, we get slightly different behaviour. We sumarise the results below:

$$\phi(t) = \left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{\frac{2}{\alpha+4}}, \tag{97}$$

$$\chi(t) = \exp\left[\frac{\beta}{\alpha}\left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{\frac{2}{\alpha+4}}\right], \tag{98}$$

$$H(t) = \left(\frac{V_0}{3}\right)^{\frac{1}{2}} \left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{-\frac{\alpha}{\alpha+4}} \exp\left[-\frac{\beta}{2}\exp\left[\frac{m\beta}{\alpha}\left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{\frac{2}{\alpha+4}}\right]\right], \tag{99}$$

$$a(t) \propto \exp\left[\frac{4+\alpha}{4}\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\left(\left(\frac{V_{0_\phi}}{3}\right)^{\frac{1}{2}}\frac{\alpha}{2}(\alpha+4)\right)^{-\frac{\alpha}{\alpha+4}} t^{\frac{4}{\alpha+4}} g_3(t)\right], \tag{100}$$

where

$$g_3(t) = \exp\left[-\frac{\beta}{2}\exp\left[\frac{m\beta}{\alpha}\left[\left(\frac{V_0}{3}\right)^{\frac{1}{2}} \frac{\alpha}{2}(\alpha+4)t\right]^{\frac{2}{\alpha+4}}\right]\right]. \tag{101}$$

D. Summary for the Coupled Potential

The summary of the inflationary behaviour is effectively equivalent to Table I with the obvious changes to the parameters. We are able to generate inflation at late, early or all times by tuning the parameters in the potential. If $m > 0$, the exponential part is dominant and the solutions produced are similar to that of a single scalar, χ with the second field, ϕ, in the background. When $m \leq 0$, the exponential part is subdominant and the results resemble those for a single scalar, ϕ, with a $\phi^{-\alpha}$ potential. The quanitative modifications are small and quickly decrease with time.

In all cases one of the fields provides the dominant contribution to $H(t)$ and $a(t)$. The effects of the second field are present but decay quickly so that the field is only in the background.

V. COSMOLOGICAL PERTURBATIONS

We now turn our attention to the evolution for cosmological perturbations in both of the scalar fields and the spacetime metric. Again, we assume that the fields are slow–rolling, implying that the parameters (3) are small. A convenient formalism to study both adiabatic and isocurvature perturbations in inflation was presented in [6], which we will use here. Instead of working with the fields ϕ and χ it is useful to perform a rotation as follows:

$$\delta\sigma = (\cos\theta)\delta\phi + (\sin\theta)\delta\chi \tag{102}$$

$$\delta s = -(\sin\theta)\delta\phi + (\cos\theta)\delta\chi, \tag{103}$$

with

$$\cos\theta = \frac{\dot\phi}{\sqrt{\dot\phi^2+\dot\chi^2}}, \qquad \sin\theta = \frac{\dot\chi}{\sqrt{\dot\phi^2+\dot\chi^2}}. \tag{104}$$

σ is called the adiabatic field and s is called the entropy field. The motivation for their names becomes clear when one considers fluctuations of them.

The line–element for arbitrary scalar perturbations of the Robertson–Walker metric for a spatially flat universe reads, (using the notation of [6])

$$\begin{aligned} ds^2 &= -(1+2A)dt^2 + 2a^2 B_{,i} dx^i dt \\ &+ a^2 \left[(1-2\psi)\delta_{ij} + 2E_{,ij} \right] dx^i dx^j . \end{aligned} \tag{105}$$

The gauge–invariant curvature perturbation, defined as

$$\mathcal{R} = \psi + \frac{H\delta\rho}{\dot\rho}, \tag{106}$$

is, on very large scales, constant for purely adiabatic perturbations (see e.g. [20] and references therein). However, entropy perturbations are a source for the curvature perturbation (106). The entropy perturbation between two species A and B, defined as

$$\mathcal{S} = \frac{\delta n_A}{n_A} - \frac{\delta n_B}{n_B}, \tag{107}$$

where n_i are the number densities of the particle species i can evolve in time, even on superhorizon scales. Therefore, it was argued [19] that on very large scales in general we have the following equations describing the evolution of $\mathcal{R}$ and $\mathcal{S}$:

$$\dot{\mathcal{R}} = \gamma H \mathcal{S}, \tag{108}$$

$$\dot{\mathcal{S}} = \delta H \mathcal{S}. \tag{109}$$

For the case of the two slow–rolling scalar fields and in the spatial flat gauge ($\psi = 0$), $\mathcal{R}$ and $\mathcal{S}$ are given by[2]

$$\mathcal{R} \approx \frac{H\left(\dot\phi\delta\phi + \dot\chi\delta\chi\right)}{\dot\phi^2 + \dot\chi^2} = \frac{H\delta\sigma}{\dot\sigma} \tag{110}$$

and

$$\mathcal{S} = \frac{H\left(\dot\phi\delta\chi - \dot\chi\delta\phi\right)}{\dot\phi^2 + \dot\chi^2} = \frac{H\delta s}{\dot\sigma}. \tag{111}$$

Fluctuations in the field σ are adiabatic perturbations, whereas fluctuations in s are entropic perturbations. On very large scales ($k \ll aH$) and in flat gauge, the evolution of fluctuations are described as [6]

$$(\delta\sigma)^{\cdot\cdot} + 3H(\delta\sigma)^{\cdot} + \left(V_{\sigma\sigma} - \dot\theta^2\right)\delta\sigma = -2V_\sigma A + \dot\sigma\dot A + 2(\dot\theta\delta s)^{\cdot} - 2\frac{V_\sigma}{\dot\sigma}\dot\theta\delta s \tag{112}$$

and

$$(\delta s)^{\cdot\cdot} + 3H(\delta s)^{\cdot} + \left(V_{ss} - \dot\theta^2\right)\delta s = -2\frac{\dot\theta}{\dot\sigma}\left[\dot\sigma((\delta\sigma)^{\cdot} - \dot\sigma A) - \ddot\sigma\delta\sigma\right]. \tag{113}$$

The metric perturbation A can be obtained from Einstein's equation and is given, in the flat gauge, by

$$HA = 4\pi G\left(\dot\phi\delta\phi + \dot\chi\delta\chi\right). \tag{114}$$

An important point is that $\dot\theta$ must be nonvanishing in order for the adiabatic field to be sourced by the entropy field. If θ is constant, the entropy field does not contribute to perturbations in the gravitational potential.

In [19] the formalism presented was applied to slow–roll inflation with two scalar fields and it was shown that γ and δ in (108) and (109) are given in terms of the slow–roll parameters

[2]For $\mathcal{S}$ we use the same normalization as in [19].

$$\gamma = -2\eta_{\sigma s}, \tag{115}$$
$$\delta = -2\epsilon + \eta_{\sigma\sigma} - \eta_{ss}, \tag{116}$$

and are, therefore, specified by the potential $V(\phi, \chi)$. In the last two equations, the slow–roll parameter are constructed from the usual slow–roll parameters for ϕ and χ (3) and are given by

$$\epsilon = \frac{1}{2}\left(\frac{V_\sigma}{V}\right)^2 \approx \epsilon_\phi + \epsilon_\chi. \tag{117}$$

and

$$\begin{aligned}
\eta_{\sigma\sigma} &= \eta_{\phi\phi}\cos^2\theta + 2\eta_{\phi\chi}\cos\theta\sin\theta + \eta_{\chi\chi}\sin^2\theta, \\
\eta_{ss} &= \eta_{\phi\phi}\sin^2\theta - 2\eta_{\phi\chi}\cos\theta\sin\theta + \eta_{\chi\chi}\cos^2\theta, \\
\eta_{\sigma s} &= (\eta_{\chi\chi} - \eta_{\phi\phi})\sin\theta\cos\theta + \eta_{\phi\chi}(\cos^2\theta - \sin^2\theta).
\end{aligned} \tag{118}$$

The time evolution of $\mathcal{R}$ and $\mathcal{S}$ between horizon crossing and some later time is given by

$$\begin{pmatrix} \mathcal{R} \\ \mathcal{S} \end{pmatrix} = \begin{pmatrix} 1 & T_{\mathcal{RS}} \\ 0 & T_{\mathcal{SS}} \end{pmatrix} \begin{pmatrix} \mathcal{R} \\ \mathcal{S} \end{pmatrix}_*, \tag{119}$$

where the asterisk marks the time of horizon crossing. The transfer functions $T_{\mathcal{RS}}$ and $T_{\mathcal{SS}}$ are given by

$$\begin{aligned}
T_{\mathcal{SS}}(t, t_*) &= \exp\left(\int_{t_*}^{t} \delta(t')H(t')dt'\right) \\
T_{\mathcal{RS}}(t, t_*) &= \int_{t_*}^{t} \gamma(t')T_{\mathcal{SS}}(t_*, t')H(t')dt'.
\end{aligned} \tag{120}$$

Thus, in the case of two slow–rolling scalar fields, the transfer functions are completely specified by the potential through the slow–roll parameters.

We are now in a position to calculate some of these transfer functions for our general potentials. We shall consider the most simple cases only in order to show how the background solutions derived in Section 3 and Section 4 can be used in order to study perturbations.

A. Two Exponentials

Let us now take

$$V(\phi, \chi) = V_{0_\phi} e^{-\beta_1 \phi} + V_{0_\chi} e^{-\beta_2 \chi}. \tag{121}$$

In this instance we find that

$$\cos\theta = \frac{\beta_2}{\sqrt{\beta_1^2 + \beta_2^2}}, \quad \sin\theta = \frac{\beta_1}{\sqrt{\beta_1^2 + \beta_2^2}}. \tag{122}$$

This shows that we get a constant angle in the phase plane. A quick glance at equations (112-113) reveals that the adiabatic and entropy perurbations decouple in this case. We should expect our analysis to reflect this. It is also straight forward to show that

$$\epsilon = \frac{1}{2}(\beta_1^2 + \beta_2^2), \tag{123}$$

$$\eta_{\sigma\sigma} = \frac{2\beta_1^2\beta_2^2}{\beta_1^2 + \beta_2^2}, \quad \eta_{ss} = \frac{\beta_1^4 + \beta_2^4}{\beta_1^2 + \beta_2^2}, \quad \eta_{\sigma s} = \frac{(\beta_2^2 - \beta_1^2)\beta_1\beta_2}{\beta_1^2 + \beta_2^2}. \tag{124}$$

The functions $\gamma(t), \delta(t)$ then turn out to be constant given by

$$\gamma = -\frac{2(\beta_2^2 - \beta_1^2)\beta_1\beta_2}{\beta_1^2 + \beta_2^2}, \tag{125}$$

$$\delta = -\frac{2(\beta_1^4 + \beta_2^2)}{\beta_1^2 + \beta_2^2}. \tag{126}$$

Then we simply compute the transfer functions using (120) to see that

$$T_{\mathcal{SS}}(t,t_*) = \left(\frac{a(t)}{a(t_*)}\right)^{\delta}, \tag{127}$$

$$T_{\mathcal{RS}}(t,t_*) = \frac{\gamma}{\delta a^{\delta}(t_*)}\left[a^{\delta}(t) - a^{\delta}(t_*)\right]. \tag{128}$$

Now since $\delta < 0$, it is immediately obvious that $T_{\mathcal{SS}}(t,t_*) \to 0$ as time passes, provided $t > t_*$. Secondly, $T_{\mathcal{RS}}(t,t_*) \to \frac{(\beta_1^2-\beta_2^2)\beta_1\beta_2}{\beta_1^4+\beta_2^4}$. This means that at late enough times $\mathcal{R}$ remains constant and $\mathcal{S} \to 0$ from (119). Therefore, in this case we only expect adiabatic perturbations and no entropy perturbations if inflation lasts long enough. This is in agreement with the results in [21].

B. A coupled example

The next simplest case will prove to be that with a coupled potential with a straight exponential. That is

$$V(\phi,\chi) = V_0\phi^{-\alpha}e^{-\beta\chi}. \tag{129}$$

We shall find it most convenient to work in terms of the field ϕ before substituting in its solution (81) to compute our integrals. We find that our θ now evolves according to

$$\cos\theta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2\phi^2}}, \qquad \sin\theta = \frac{\beta\phi}{\sqrt{\alpha^2+\beta^2\phi^2}}. \tag{130}$$

We also have the slow–roll parameters

$$\epsilon = \frac{1}{2}\left[\frac{\alpha^2}{\phi^2} + \beta^2\right], \tag{131}$$

$$\eta_{\phi\phi} = \frac{\alpha(\alpha+1)}{\phi^2}, \qquad \eta_{\phi\chi} = \frac{\alpha\beta}{\phi}, \qquad \eta_{\chi\chi} = \beta^2. \tag{132}$$

It is then straightforward to show that

$$\eta_{\sigma\sigma} = \frac{\alpha^2}{\alpha^2+\beta^2\phi^2}\left[\frac{\alpha(\alpha+1)}{\phi^2} + 2\beta^2 + \frac{\beta^4}{\alpha^2}\phi^2\right], \tag{133}$$

$$\eta_{ss} = \frac{\alpha^2}{\alpha^2+\beta^2\phi^2}\frac{\beta^2}{\alpha}, \qquad \eta_{\sigma s} = -\frac{\alpha^2}{\alpha^2+\beta^2\phi^2}\frac{\beta}{\phi}. \tag{134}$$

We then find that the transfer parameters are given as

$$\gamma(t) = \frac{2\beta\alpha^2}{\phi(\alpha^2+\beta^2\phi^2)} = \beta^2\alpha^{\frac{1}{2}}\frac{1}{(\ln t)^{\frac{1}{2}}(\alpha+4\ln t)}, \tag{135}$$

$$\delta(t) = \frac{\alpha^2}{\alpha^2+\beta^2\phi^2}\left[\frac{\alpha}{\phi^2} - \frac{\beta^2}{\alpha}\right] = \frac{\beta^2(\alpha-4\ln t)}{4(\ln t)(\alpha+4\ln t)}. \tag{136}$$

We have also verified these relations numerically. Now it remains to compute the transfer functions which we are able to do exactly. From (83) we see that

$$H(t) = \frac{2}{\beta^2 t}. \tag{137}$$

This then gives

$$T_{\mathcal{SS}}(t,t_*) = \exp\left(I(t,t_*)\right), \tag{138}$$

where

$$I(t,t_*) = \int_{t_*}^{t} \frac{(\alpha - 4\ln t')}{2(\ln t')(\alpha + 4\ln t')}\frac{1}{t'}\, dt' \tag{139}$$

$$= \frac{1}{2}\ln\left[\frac{\ln t}{\ln t_*}\right] - \ln\left[\frac{\alpha + 4\ln t}{\alpha + 4\ln t_*}\right]. \tag{140}$$

Plugging this back in we find that

$$T_{\mathcal{SS}}(t,t_*) = \left[\frac{\ln t}{\ln t_*}\right]^{\frac{1}{2}}\left[\frac{\alpha + 4\ln t_*}{\alpha + 4\ln t}\right]. \tag{141}$$

Then using this result

$$T_{\mathcal{RS}}(t,t_*) = \beta^2\alpha^{\frac{1}{2}}\frac{\alpha + 4\ln t_*}{(\ln t_*)^{\frac{1}{2}}}\int_{t_*}^{t}\frac{2}{\beta^2 t'}\frac{1}{(\alpha + 4\ln t')^2}\, dt' \tag{142}$$

$$= 2\alpha^{\frac{1}{2}}\frac{\alpha + 4\ln t_*}{(\ln t_*)^{\frac{1}{2}}}\left[-\frac{1}{4}\frac{1}{\alpha + 4\ln t'}\right]_{t_*}^{t}, \tag{143}$$

$$= \frac{\alpha^{\frac{1}{2}}}{2}\left(\frac{1}{(\ln t_*)^{\frac{1}{2}}} - \frac{\alpha + 4\ln t_*}{(\ln t_*)^{\frac{1}{2}}}\frac{1}{\alpha + 4\ln t}\right). \tag{144}$$

Thus at late times we see

$$T_{\mathcal{SS}} \to 0, \quad T_{\mathcal{RS}} \to \frac{\alpha^{\frac{1}{2}}}{2}\frac{1}{(\ln t_*)^{\frac{1}{2}}}. \tag{145}$$

Again, this implies that at late times $\mathcal{R}$ remains constant, although its value on a certain length scale depends on the time of horizon crossing t_*.

VI. CONCLUSIONS

In this paper we have studied inflationary solutions generated by two scalar fields and potentials of the form $V = V_0\phi^{-\alpha}\exp(-\beta\phi^m)$. This potential is very general and encompasses potentials motivated by supergravity and string theory.

We have seen that it is possible to generate a large range of different inflating universes by varying the parameters in the theory. We find that slow–roll is a valid approximation by means of numerical calculations and we are able to verify that the analytic solutions presented in this paper hold. For two uncoupled fields (Section 3), we found that, often, only one of the fields is dominant. However, we also found solutions where both fields are important for the inflationary dynamics. For two coupled fields with the above potential (Section 4), we have found that only one of the fields dominates inflation, whereas the other field is only a background field. This case might be interesting if the background field acted as a curvaton field, for example.

We have used some of the background solutions we found to describe the evolution of the perturbations in two specific cases and, potentially, one would be able to find the transfer functions for all the solutions discussed here. We have demonstrated this with two examples. The solutions presented here thus allow one to make predictions about the relative size of $\mathcal{R}$ and $\mathcal{S}$ for a large class of inflationary scenarios. In order to make predictions for the anisotropies in the cosmic microwave background radiation, one has to follow the perturbations into the radiation dominated era, which involves a detailed calculation of the decay of both fields. Ending inflation in the models discussed here will involve presumably a third field, which becomes important just at the end of the inflationary stage.

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